Periodic orbits of Mobius functions

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Abstract

The purpose of this article is to find conditions of existence of n-periodic orbits for Mobius functions and determine all such orbits (in the case of their existence).

Part1.

We start with concrete problem.

Problem.(Dutch Mathematical Olympiad,1983 and Math Excalibur Vol.1,No.4, Problem 16)

Let a, b, c be real numbers , with a, b, c not equal, such that

$$a + \frac{1}{b} = t, b + \frac{1}{c} = t, c + \frac{1}{a} = t.$$

Determine all possible value of t and prove that $abc + t = 0$.

Solution.

Obvious that $a, b, c \notin \{0, t\}$. Also note that $t \neq 0$,because otherwise $ab = bc = ca = -1$ implies $a^2b^2c^2 = -1$.

Since $a, b, c \notin \{0, t\}$ then

$$\textbf{(1)} \quad \begin{cases} a + \dfrac{1}{b} = t \\ b + \dfrac{1}{c} = t \\ c + \dfrac{1}{a} = t \end{cases} \iff \begin{cases} b = \dfrac{1}{t-a} \\ c = \dfrac{1}{t-b} \\ a = \dfrac{1}{t-c} \end{cases} \iff \begin{cases} b = h(a) \\ c = h(b) \\ a = h((c)) \end{cases},$$

where $h(x) := \dfrac{1}{t-x}$ for any $x \in \mathbb{R} \diagdown \{0, t\}$.

We can see that for $x \in \{a, b, c\}$ holds $x = h(h(h(x)))$, .

that is function $h(h(h(x)))$ have three distinct fixed points.

Since for $x \in \{a, b, c\}$ we have $h(h(h(x))) = \dfrac{1}{t - \dfrac{1}{t - \dfrac{1}{t-x}}} = \dfrac{t^2 - tx - 1}{t^3 - t^2x - 2t + x}$

then $h(h(h(x))) = x \iff \dfrac{t^2 - tx - 1}{t^3 - t^2x - 2t + x} = x \iff$

$t^3x - t^2x^2 - 2tx + x^2 = t^2 - tx - 1 \iff \left(1 - t^2\right)\left(x^2 - xt + 1\right) = 0$

implies $t^2 = 1$, because otherwise quadratic equation $x^2 - xt + 1 = 0$ have three distinct roots a, b and c, that is a contradiction.

Let $t^2 = 1$.

Then $h(x) \neq t$ for any $x \in \mathbb{R}\diagdown\{0,t\}$ (because $h(x) = t \iff \dfrac{1}{t-x} = t \iff x = \dfrac{t^2-1}{t} = 0$) and, therefore,

$h : \mathbb{R}\diagdown\{0,t\} \to \mathbb{R}\diagdown\{0,t\}$.

Also for any $x \in \mathbb{R}\diagdown\{0,t\}$ we have

$h(h(x)) = \dfrac{1}{t - \dfrac{1}{t-x}} = \dfrac{t-x}{t^2-tx-1} = \dfrac{t-x}{-tx}$ and

$h(h(h(x))) = \dfrac{t^2-1-tx}{t^3-t^2x-2t+x} = \dfrac{-tx}{t-x-2t+x} = x$,

that is any $x \in \mathbb{R}\diagdown\{0,t\}$ is fixed point for $h \circ h \circ h$.

Noting that $h(x) \neq x$ and $h(h(x)) \neq x$ for any $x \in \mathbb{R}\diagdown\{0,t\}$ because $h(x) = x \iff x^2-tx+1=0$ and $h(h(x)) = x \iff tx^2-x+t=0 \iff x^2-tx+1=0$, where equation $x^2-tx+1=0$ have no solutions in $\mathbb{R}$ we can conclude that set of all triples of real numbers (a,b,c) such that a,b,c are distinct and satisfies **(1)** can be parameterized by $x \in \mathbb{R}\diagdown\{0,t\}$ as follows

$(a,b,c) = \left(x, \dfrac{1}{t-x}, \dfrac{x-t}{tx}\right)$.

Thus, $t^2 = 1$ and $abc = x \cdot \dfrac{1}{t-x} \cdot \dfrac{x-t}{tx} = -t \iff abc + t = 0$.

Part 2. Terminology and notations.

In order to move forward we need to make some preparation.
Let $f(x)$ be function with domain $D \subset \mathbb{R}$ such that $f : D \longrightarrow D$.
For any $x \in D$ we will consider the sequence $(x_n)_{n\geq 0}$ defined recursively as follows:

$x_0 := x, x_1 := f(x_0)$,and for any $n \in \mathbb{N}$ if $x_n \in D$ then $x_{n+1} := f(x_n)$.

Such sequence, infinite or finite, we call orbit of x created by f and denote $\mathcal{O}_f(x)$ or simpler $\mathcal{O}(x)$.
If $x_n \in D$ for any $n \in \mathbb{N}$ then orbit $\mathcal{O}_f(x)$ is infinite, otherwise orbit is finite.
Let function f_0 be defined by $f_0(x) = x$ and for any natural n we define recursively n-iterated function f_n by
$f_n = f \circ f_{n-1}, n \in \mathbb{N}$, that is $f_1(x) := f(x)$ and $f_1(x) := f(f_n(x))$ for any $x \in D$.Thus, $x_n = f_n(x), n \in \mathbb{N}$.
Using Math Induction we can prove that $f_n \circ f_m = f_{n+m}$ for any $n, m \in \mathbb{N}$.
Indeed, for any $n \in \mathbb{N}$, assuming $f_n \circ f_m = f_{n+m}$ we obtain
$f_{n+1} \circ f_m = (f \circ f_n) \circ f_m = f \circ (f_n \circ f_m) = f \circ f_{n+m} = f_{n+1+m}$.
By the way we obtain $f_n \circ f_m = f_{n+m} = f_{m+n} = f_m \circ f_n$ (although, the operation of the composition is generally non-commutative).
Let $x \in D$ be number such that $x_m = x \iff f_m(x) = x$ for some $m \in \mathbb{N}$ then point x (which is fixed point of f_m) we also call periodic.
Then orbit $\mathcal{O}_f(x)$ is periodic orbit and, of course, infinite.
In that case the smallest natural n such that $x_n = x$ we will call

main period of x and denote $\mu(x)$.
Also if $\mu(x) = n$ then correspondent orbit $\mathcal{O}_f(x)$ and point x we call n-periodic. (Obvious that any period m is multiples of the main period n, because if $m = kn + r$,where remainder $r \neq 0$ then $x = f_n(x) = f_{kn+r}(x) = (f_{kn} \circ f_r)(x) = f_r(x)$ .Since $r < n = \mu(x)$ then it is the contradiction).
If $\mathcal{O}_f(x)$ is periodic orbit with $\mu(x) = n$ then x is fixed point for function f_n, that is solution of equation $f_n(x) = x$.
Thus, point x is n-periodic of the following conditions are satisfied:
1. $f_k(x) \in D, k = 1, 2, ..., n-1$;
2. $f_k(x) \neq x, k = 1, 2, ..., n-1$;
3. $f_n(x) = x$.
Let D_∞ be subset of all $x \in D$ for which f generate infinite orbit.
If D_∞ is non empty then restriction f on D_∞ give us mapping
$f : D_\infty \longrightarrow D_\infty$.
Indeed, if $x \in D_\infty$ that is $\mathcal{O}(x)$ is infinite then $\mathcal{O}(f(x))$ is subsequence of $\mathcal{O}(x)$ and infinite as well.
Periodic orbit $\mathcal{O}(x)$ with $\mu(x) = n$ such that $x_0, x_1, ..., x_{n-1}$ not equal we will call strictly periodic.
Applying this terminology to the problem, solved above, we can formulate the following
Theorem.
Function $x \longmapsto h(x) = \dfrac{1}{t-x} : \mathbb{R} \diagdown \{t\} \longrightarrow \mathbb{R}$ have strictly periodic orbit $\mathcal{O}_h(x)$ with main period 3 if and only if $t^2 = 1$.
In that case for any $x \in \mathbb{R} \diagdown \{0, t\}$ orbit $\mathcal{O}_h(x)$ is strictly periodic with $\mu(x) = 3$ and $xh_1(x)h_2(x) + t = 0$.

Part 3. Generalization and modification

Generalization.

Let now n be any natural number and let $\mathcal{T}_n$ be set of all real t such that function $h(x) = \dfrac{1}{t-x}$ have periodic orbits of main period n.
We already know that $\mathcal{T}_3 = \{-1, 1\}$. And we going to find $\mathcal{T}_n$ effectively, find its explicit representation for all other n, but first we will find $\mathcal{T}_1$ and $\mathcal{T}_2$
1. Let $n = 1$, then

$$h(x) = x \iff x = \frac{1}{t-x} \iff xt - x^2 = 1 \iff x^2 - xt + 1 = 0.$$

Thus we obtain that if h has fixed point x,or by the other words has orbit with the period 1 then $t^2 - 4 \geq 0 \iff |t| \geq 2$.
Let $|t| \geq 2$. For each t such that $|t| > 2$ we have two fixed points of h namely, solutions x_1, x_2 of equation $x^2 - xt + 1 = 0$ and, respectively, two infinite orbits
$\mathcal{O}_h(x) = (x, x, ..., x, ...)$, $x \in \{x_1, x_2\}$
and one infinite orbit

$O_h\left(\dfrac{t}{2}\right)=\left\{\dfrac{t}{2},\dfrac{t}{2},...,\dfrac{t}{2},..\right\}$ for each $t\in\{-2,2\}$.
Thus $\mathcal{T}_1=(-\infty,2]\cup[2,\infty)$.
Remark.
It is not difficult to prove that in case $|t|=2$ any $x\neq\dfrac{t}{2}$
generate infinite non-periodic orbit.
For example if $t=2$ then we have

$$\mathcal{O}_h(x)=\left(x,\frac{1}{2-x},\frac{2-x}{3-2x},...,\frac{n-(n-1)x}{n+1-nx},...\right)$$

if $x\neq 1$ and further we will see that in the case $|t|>2$ orbit
$\mathcal{O}_h(x)$ is infinite and non-periodic for any $x\neq x_1,x_2$.

2. Let $n=2$ and let $\mathcal{O}_h(x)$ is periodical orbit with $\mu(x)=2$.
Then

$$h(h(x))=x\iff x=\frac{1}{t-\dfrac{1}{t-x}}=\frac{t-x}{t^2-tx-1}\iff$$

$t^2x-tx^2-x=t-x\iff t(x^2-xt+1)=0\iff t=0,$
since $x^2-xt+1\neq 0$.Thus $\mathcal{T}_2=\{0\}$.
Let $t=0,$ then any point $x\neq 0$ generate periodical orbit
$\mathcal{O}(x)=\left(x,-\dfrac{1}{x},x,-\dfrac{1}{x},...\right)$ with $\mu(x)=2$.

3. Let now $n\geq 2$ be any and let $\mathcal{O}_h(x)$ is periodical orbite
with $\mu(x)=n$.It is mean that for $x\in\mathbb{R}\setminus\{0,t\}$, which generate
this orbit, holds $h_1(x),...,h_{n-1}(x)\neq x,t$ and $h_n(x)=x$.
First note that $g(y):=\dfrac{ty-1}{y}:\mathbb{R}\setminus\{0,t\}\longrightarrow\mathbb{R}\setminus\{0,t\}$ is inverse to h,
that is $h(g(y))=y$,for any $y\neq 0,g(y)\neq t$ and $g(h(x))=x$ for any
$x\neq t,h(x)\neq 0$.
Also note that if $\mathcal{O}_h(x)$ be periodic orbit with $\mu(x)=n$ then numbers
$x,h_1(x),...,h_{n-1}(x)$ all different.
Indeed, assume that there are $0\leq i<j\leq n-1$ such that $h_i(x)=h_j(x)$.
If $i=0$ then $x=h_j(x)$ contradict to $x\neq h_k(x)$ for any $k=1,...,n-1$;
if $i>0$ then applying g we obtain

$$h_i(x)=h_j(x)\iff g(h_i(x))=g(h_j(x))\iff h_{i-1}(x)=h_{j-1}(x)\iff ...\iff x=h_{j-1}(x)$$

that is contradiction as well.
So, further we don't need to claim that numbers $x,h_1(x),...,h_{n-1}(x)$
all different.
Enough to claim that $h_k(x)\neq t,k=1,2,...,n-1$.
We will prove that $h_n(x)$,which defined by recurrence
$h_n(x)=h(h_{n-1}(x)),n\in\mathbb{N}$ with $h_0(x)=x$ can be represented in the
form $h_n(x)=\dfrac{P_n(x,t)}{Q_n(x,t)}$ or shortly as $\dfrac{P_n}{Q_n}$.

Since $h_0(x) = \dfrac{x}{1}$ and $h_1(x) = \dfrac{1}{t-x}$ we claim
$P_0 = x, P_1 = 1, Q_0 = 1, Q_1 = t - x$.
Also, since $\dfrac{P_{n+1}}{Q_{n+1}} = h\left(\dfrac{P_n}{Q_n}\right) = \dfrac{1}{t - \dfrac{P_n}{Q_n}} = \dfrac{Q_n}{tQ_n - P_n}$ we claim
$P_{n+1} = Q_n$ and $Q_{n+1} = tQ_n - P_n$.
This implies $P_{n+1} = tP_n - P_{n-1}, n \in \mathbb{N}$ and $Q_n = P_{n+1}$.
Note that $P_2 = t - x$ and let $\overline{h}_n(x) := \dfrac{P_n}{P_{n+1}}, n \in \mathbb{N} \cup \{0\}$.
Since, $h_0(x) = \overline{h}_0(x), h_1(x) = \overline{h}_1(x)$ and for any $n \in \mathbb{N} \cup \{0\}$ assuming $h_n(x) = \overline{h}_n(x)$ we obtain $h_{n+1}(x) = h(h_n(x)) = h\left(\overline{h}_n(x)\right) = \overline{h}_{n+1}(x)$
then by Math Induction $h_n(x) = \overline{h}_n(x) = \dfrac{P_n}{P_{n+1}}$ for all $n \in \mathbb{N} \cup \{0\}$.
Condition $h_n(x) = x$ is equivalent to $\dfrac{P_n}{P_{n+1}} = x \iff P_n - xP_{n+1} = 0$.
Observation of cases $n = 2, 3$ lead us to assumption

$$P_n - xP_{n+1} = R_n(t)\left(x^2 - xt + 1\right)$$

where $R(t)$ is the polynomial of degree $n - 1$.
In particularly $R_2(t) = t, R_3(t) = t^2 - 1, R_4(t) = t^3 - 2t, R_5(t) = t^4 - 3t^2 + 1$.
Since $P_{n+1} - xP_{n+2} = t(P_n - xP_{n+1}) - (P_{n-1} - xP_n) \iff R_{n+1}(t)\left(x^2 - xt + 1\right) = \left(x^2 - xt + 1\right)(tR_n(t) - R_{n-1}(t))$ and $x^2 - xt + 1 \neq 0$ (because $n \geq 2$)
we obtain for $R_n(t)$ recurrence

(2) $\qquad R_{n+1}(t) = tR_n(t) - R_{n-1}(t), n \geq 2$

with initial condition $R_1(t) = 1, R_2(t) = t$.($R_0 := 0$).
Suppose on a while that $|t| < 2$ (this restriction on t isn't influence on definition of the polynomial).
Then for $\varphi := \cos^{-1}\left(\dfrac{t}{2}\right)$ we have $t = 2\cos\varphi, t^2 - 2 = 2\cos 2\varphi$ and recurrence (1) can be rewritten in the form

$$R_{n+1} = 2\cos\varphi R_n - R_{n-1},$$

Since $R_n = c_1 \cos n\varphi + c_2 \sin n\varphi$ and from $R_0 = 0, R_1 = 1$ follows $c_1 = 0$, $1 = c_2 \sin\varphi \iff c_2 = \dfrac{1}{\sin\varphi}$ then we obtain

$$R_n = R_n(2\cos\varphi) = \frac{\sin n\varphi}{\sin\varphi} \text{ and } R_n(t) = \frac{\sin\left(n \cdot \cos^{-1}\left(\frac{t}{2}\right)\right)}{\sin\left(\cos^{-1}\left(\frac{t}{2}\right)\right)}).$$

Let $T_n(x)$ be Chebishev Polynomial of the First Kind defined by

$$T_n(\cos\varphi) = \cos n\varphi,$$

or, by recurrence $T_{n+1} - 2xT_n + T_{n-1} = 0, n \in \mathbb{N}$ and $T_0 = 1, T_1 = x$.
We have $(T_n(\cos\varphi))' = T_n'(\cos\varphi)(-\sin\varphi) = -n\sin n\varphi \implies$

$$T_n'(\cos\varphi) = \frac{n\sin n\varphi}{\sin\varphi}.$$

Polynomial $U_{n-1}(x) = \dfrac{T_n'(x)}{n}$ degree $n - 1$ we call Chebishev Polynomial

of the Second Kind.
$U_n(x)$ satisfy to recurrence $U_{n+1} = 2xU_n - U_{n-1},\ n \in \mathbb{N}$, (the same as T_n but with different initial conditions: $U_0 = 1, U_1 = 2x$).
Since $U_{n-1}(t) = \dfrac{\sin\left(n \cdot \cos^{-1}(t)\right)}{\sin\left(\cos^{-1}(t)\right)}$ and $U_{n+1} = 2tU_n - U_{n-1}, n \in \mathbb{N}$, with $U_0 = 1, U_1 = 2t$ and $R_{n+2}(x) = tR_{n+1}(t) - R_n(t), n \in \mathbb{N}$ with $R_1(t) = 1, R_2(t) = t$ we can see that

$$R_n(t) = U_{n-1}\left(\frac{t}{2}\right).$$

Now we can find all roots of polynomial $R_n(t)$.
Since $\dfrac{\sin n\varphi}{\sin\varphi} = 0 \iff \left\{\begin{array}{c} \varphi = \dfrac{k\pi}{n} \\ \sin\varphi \neq 0 \end{array}\right| \iff \varphi = \dfrac{k\pi}{n}$ and $n \nmid k$,
we consider $n-1$ different numbers $t_k = 2\cos\dfrac{k\pi}{n},\ k = 1, 2, ..., n-1$.
Easy to see that $R_n(t_k) = R_n\left(2\cos\dfrac{k\pi}{n}\right) = \dfrac{\sin k\varphi}{\sin\dfrac{k\pi}{n}} = 0.$
So, $t_1, t_2, ..., t_{n-1}$ are $n-1$ real solution of equation $R_n(t) = 0$ and, because $\deg R_n(t) = n-1$, then $t_1, t_2, ..., t_{n-1}$ are all roots of $R_n(t)$.
But we need only such of this roots, which can't be roots of $R_m(t)$ with $m < n$. That is only k coprime with n satisfy to this claim.
(If we assume opposite that $R_m(t) = 0$ for some $m \in \{1, 2, ..., n-1\}$ then

$$R_m(t) = 0 \iff U_{m-1}\left(\frac{t}{2}\right) = 0 \iff \sin\left(m \cdot \cos^{-1}\left(\frac{t}{2}\right)\right) = 0 \iff$$

$$\sin\left(m \cdot \cos^{-1}\left(\cos\frac{k\pi}{n}\right)\right) = 0 \iff \sin\frac{mk\pi}{n} = 0 \iff$$

$$mk \text{ is divisible by } n \iff m \text{ is divisible by } n \text{ (because} \gcd(k,n) = 1).$$

That is we obtain a contradiction with $m \in \{1, 2, ..., n-1\}$).
Thus we have only $\phi(n)$ different t which provide $n-$periodic orbits, namely,

$$\mathcal{T}_n = \left\{t \mid t = 2\cos\frac{k\pi}{n}\text{, where } k = 1, 2, ..., n-1 \text{ and } \gcd(k,n) = 1\right\}.$$

In particular, if $n = 6$, then only $k = 1, 5$ are coprime with 6,hence we have $t = 2\cos\dfrac{\pi}{6} = \sqrt{3}$ and $t = 2\cos\dfrac{5\pi}{6} = -\sqrt{3}$,that is $\mathcal{T}_6 = \left\{-\sqrt{3}, \sqrt{3}\right\}$
Now for each $t \in \mathcal{T}_n$ we will find set $D_n(t)$ of all $n-$periodic x that is x with $\mu(x) = n$.
Let $t = 2\cos\dfrac{k\pi}{n}$, where $k = 1, 2, ..., n-1$ and $\gcd(k,n) = 1$.
Since $R_n(t) = 0,\ \prod\limits_{k=1}^{n-1} R_k(t) \neq 0, R_{n+1}(t) = -R_{n-1}(t) \neq 0$ and

$x_m := \dfrac{R_{m+2}(t)}{R_{m+1}(t)}, m = 0,1,2,...$
then we have
$x_0 = \dfrac{R_2(t)}{R_1(t)} = t, x_{n-2} = \dfrac{R_n(t)}{R_{n-1}(t)} = 0, x_{n-1} = \dfrac{R_{n+1}(t)}{R_n(t)} = \pm\infty.$
Since $R_p(t) = \dfrac{\sin\left(p\cdot\cos^{-1}\left(\cos\dfrac{k\pi}{n}\right)\right)}{\sin\left(\cos^{-1}\left(\cos\dfrac{k\pi}{n}\right)\right)} = \dfrac{\sin\left(\dfrac{pk\pi}{n}\right)}{\sin\left(\dfrac{k\pi}{n}\right)}$ for $k = 1,2,...,n-1$
and $\gcd(k,n) = 1, p \in \mathbb{N}$ then if $n \geq 4$ for $m = 1,2,..,n-3$
we obtain $x_m = \dfrac{R_{m+2}(t)}{R_{m+1}(t)} = \dfrac{\sin\left(\dfrac{(m+2)k\pi}{n}\right)}{\sin\left(\dfrac{(m+1)k\pi}{n}\right)}.$
Thus, for $t = 2\cos\dfrac{k\pi}{n}$ where $k = 1,2,...,n-1$ and $\gcd(k,n) = 1$ we have $D(t) = \mathbb{R}\diagdown\{t,0,x_1,...,x_{n-3}\}$ and for any $x \in D(t)$ correspondent orbit $\mathcal{O}_h(x)$ is $n-$periodic.

Remark 1.
For each $t_k = 2\cos\dfrac{k\pi}{n}$, where $k = 1,2,...,n-1$ and $k \perp n$ set $\{e, h_1, h_2, ..., h_{n-1}\}$ is a cyclic group with respect to composition as multiplication, where $h_n = h_0 = e$ and $h_k^{-1} = h_{n-k}, k = 1,...,n-1.$

Remark 2.
Since $|t_k| \leq 2$ then $R_n(t) \neq 0$ for any $n \in \mathbb{N}$ if $|t| > 2$ and if at the same time x isn't root of equation $x^2 - xt + 1 = 0$ then equation $h_n(x) = x \iff R_n(t)\left(x^2 - xt + 1\right) = 0$ have no solutions for any $n \in \mathbb{N}$ and, therefore, orbit $\mathcal{O}_h(x)$ is infinite and non-periodic

Modification.
Let's consider the similar problem with respect to function $h(x) = \dfrac{-1}{t-x}$, namely, for any $n \in \mathbb{N}$ we will find $\mathcal{T}_n$ - set of all real t such that function $h(x)$ have periodical orbits main period n.
If $n = 1$, then equation $x = \dfrac{-1}{t-x} \iff x^2 - xt - 1 = 0$ have two solutions $x_{1,2} = \dfrac{t+\sqrt{t^2+4}}{2}$ for any real t.Thus, $\mathcal{T}_1 = \mathbb{R}$ and we have two orbits
$\mathcal{O}_h(x_1) = (x_1, x_1, ...), \mathcal{O}_h(x_2) = (x_2, x_2, ...).$
Let $n = 2$.Then $x = \dfrac{-1}{t - \dfrac{-1}{t-x}} = \dfrac{x-t}{t^2 - tx + 1} = h_2(x) \iff$
$x - tx^2 + t^2x = x - t \iff t\left(x^2 - xt - 1\right) = 0$ and since $h(x) \neq x \iff x^2 - xt - 1 \neq 0$ we obtain that
$\mathcal{T}_2 = \{0\}$ and for any real $x \neq 0,1$ we have $\mathcal{O}_h(x) = \left(x, \dfrac{1}{x}, x, \dfrac{1}{x}, ...\right)$
Let $n = 3$. Since $h_2(x) \neq x$ implies $x^2 - xt - 1 = 0 \neq 0, t \neq 0$ and

$x = h_3(x) = \dfrac{-1}{t - h_2(x)} = \dfrac{-1}{t - \dfrac{x-t}{1-tx+t^2}} = \dfrac{1}{\dfrac{x-t}{1-tx+t^2} - t} \iff$

$x = \dfrac{1-tx+t^2}{x-t-t+t^2x-t^3} \iff x^2 - 2tx + t^2x^2 - xt^3 = 1 - tx + t^2 \iff$

$x^2\left(t^2+1\right) - xt\left(t^2+1\right) - \left(t^2+1\right) = 0 \iff \left(t^2+1\right)\left(x^2-xt-1\right) = 0$ then for x such that $h_i(x) \neq x, i = 1,2$ the equation $x = h_3(x)$ have no solution in real numbers.

So, function $h(x) = \dfrac{1}{x-t}$ have no 3– periodical orbits in $\mathbb{R}$ and $\mathcal{T}_3 = \varnothing$.

As above we will use representation $h_n(x) = \dfrac{P_n(x,t)}{Q_n(x,t)}$ or shortly as $\dfrac{P_n}{Q_n}$.

Since $h_0(x) = \dfrac{x}{1}$ and $h_1(x) = \dfrac{-1}{t-x}$ we have

$P_0 = x, P_1 = -1, Q_0 = 1, Q_1 = t - x.$

From $\dfrac{P_{n+1}}{Q_{n+1}} = \dfrac{-1}{t - \dfrac{P_n}{Q_n}} = \dfrac{-Q_n}{tQ_n - P_n}$ follows

$P_{n+1} = -Q_n$ and $Q_{n+1} = tQ_n - P_n.$

This imply $P_{n+1} = tP_n + P_{n-1}$ and $Q_n = -P_{n+1}$.

Condition $h_n(x) = x$ equivalent to $-\dfrac{P_n}{P_{n+1}} = x \iff P_n + xP_{n+1} = 0.$

Observation of cases $n = 1,2,3$ lead us to assumption $h_n(x) = x \iff$

$P_n + xP_{n+1} = R_n(t)\left(x^2 - xt - 1\right)$

where $R_n(t)$ is the polynomial degree $n-1$.

In particular $R_2(t) = t, R_3(t) = t^2 + 1$.

Let there is orbit with main period $n > 1$.Since $x^2 - xt + 1 \neq 0$ (because otherwise we have periodical orbit with main 1) then

$P_{n+1} + xP_{n+2} = t\left(P_n + xP_{n+1}\right) + P_{n-1} + xP_n \iff$

$tR_{n+1}(t)\left(x^2 - xt - 1\right) + R_n(t)\left(x^2 - xt + 1\right) + R_{n-1}(t)\left(x^2 - xt + 1\right) \iff$

$\left(x^2 - xt - 1\right)\left(R_{n+1}(t) - tR_n(t) - R_{n-1}(t)\right) = 0$

and we obtain for $R_n(x)$ recurrence

(3) $R_{n+1}(x) = tR_n(t) + R_{n-1}(t)$ with initial condition

$R_1(t) = 1, R_2(t) = t.$

Therefore, $h_n(x) = x \iff P_n + xP_{n+1} = 0 \iff$

$R_n(t)\left(x^2 - xt - 1\right) = 0 \iff R_n(t) = 0$ since $x^2 - xt + 1 \neq 0$.

We will prove, that for any $n > 2$ equation $R_n(t) = 0$ have no nonzero solutions.

(case $t = 0$ (there is 2-periodical orbit) must be excluded).

Because situation is different for n odd and n even we will consider separately polynomials $R_{2n+1}(t)$ and polynomials

$\bar{R}_{2n}(t) = \dfrac{R_{2n}(t)}{t}.$

Since $R_{n+2} = tR_{n+1} + R_n = t\left(tR_n + R_{n-1}\right) + R_n =$ $\left(t^2+1\right)R_n + tR_{n-1}$ and $tR_{n-1} = R_n - R_{n-2}$ we obtain

$R_{n+2} = \left(t^2+2\right)R_n - R_{n-2}.$

Thus we consider two sequences:

$\bar{R}_{2n}(t), n \in \mathbb{N} \cup \{0\}$, which satisfy $\bar{R}_{2n+2} = (t^2+2)\bar{R}_{2n} - \bar{R}_{2n-2},\ n \geq 1$ with $\bar{R}_0 = 0, \bar{R}_2 = 1$ and $R_{2n-1}(t), n \in \mathbb{N}$,which satisfy
$R_{2n+3} = (t^2+2) R_{2n+1} - R_{2n-1},\ n \geq 1$ and $R_1 = 1, R_3 = t^2+1$.

Lemma.
For all $n \in \mathbb{N}$ holds:
i. $R_{2n+1} > R_{2n-1} > 0$;
ii. $\bar{R}_{2n+2} > \bar{R}_{2n} > 0$.
Proof.(by Math. Induction)
1.Base of induction.
Let $n = 1$,then $R_3 = t^2+1 > 1 = R_1 > 0$ and $\bar{R}_4 = t^2+2 > 1 = \bar{R}_2 > 0$.
2.Step of induction.
i. Let $R_{2n+1} > R_{2n-1} > 0$, then
$R_{2n+3} - R_{2n+1} = (t^2+1) R_{2n+1} - R_{2n-1} > R_{2n+1} - R_{2n-1} > 0$,
so, $R_{2n+3} > R_{2n+1} > 0$;
ii. Let $\bar{R}_{2n+2} > \bar{R}_{2n} > 0$,then
$\bar{R}_{2n+4} - \bar{R}_{2n+2} = (t^2+1)\bar{R}_{2n+2} - \bar{R}_{2n} > \bar{R}_{2n+2} - \bar{R}_{2n} > 0$,
so, $\bar{R}_{2n+4} > \bar{R}_{2n+2} > 0$.
Alternative proof.
Since characteristic equation $x^2 - tx - 1 = 0$ for recurrence **(3)**
have roots $x_1 = \dfrac{t - \sqrt{t^2+4}}{2} < 0$, $x_2 = \dfrac{t + \sqrt{t^2+4}}{2}$ with Vieta's properties $x_1 + x_2 = t$ and $x_1 x_2 = -1$
then $R_n = c_1 x_1^n + c_2 x_2^n$,where c_1, c_2 can be determined from initial conditions $R_0 = 0, R_1 = 0$.
Since $c_1 = -\dfrac{1}{\sqrt{t^2+4}}$,$c_2 = -\dfrac{1}{\sqrt{t^2+4}}$ then, $R_n = \dfrac{x_2^n - x_1^n}{x_2 - x_1}$
For odd n we have $R_n = \dfrac{x_2^n - x_1^n}{x_2 - x_1} = \dfrac{x_2^n + (-x_1)^n}{x_2 - x_1} > 0$.
For $n = 2m$ we have
$R_{2m} = \dfrac{x_2^{2m} - x_1^{2m}}{x_2 - x_1} = (x_2 + x_1)\left(x_2^{2m-2} + x_1^{2m-4} x_2^2 + ... + x_1^{2m-2}\right) =$
$t\left(x_2^{2m-2} + x_1^{2m-4} x_2^2 + ... + x_1^{2m-2}\right)$.
Thus $\bar{R}_{2m} = \dfrac{R_{2m}}{t} = x_2^{2m-2} + x_1^{2m-4} x_2^2 + ... + x_1^{2m-2} > 0$.
Corollary.
From lemma immediately follows that $R_n(t)$ have no nonzero roots.
So function $h(x) = \dfrac{-1}{t-x}$ have no $n-$periodical orbits with $n > 2$.

Part 4 More generalization
Now we will show that the general problem about periodicity of orbits for any Möbius Function $g(x) = \dfrac{ax+b}{cx+d}$ (where a, b, c, d satisfy to $ad - bc \neq 0$ and $c \neq 0$) can be reduced to the considered above two cases.
First note, that for any linear function $l(x) = px + q, p \neq 0$ orbits

of element $x \in \mathbb{R}$ for Mobius Functions g and $f = l^{-1} \circ g \circ l$
have the same periodicity.
Indeed, we have
$h_2 = \left(l^{-1} \circ g \circ l\right) \circ \left(l^{-1} \circ g \circ l\right) = \left(l^{-1} \circ g\right) \circ \left(l \circ l^{-1}\right) \circ (g \circ l) =$
$\left(l^{-1} \circ g\right) \circ (g \circ l) = l^{-1} \circ (g \circ g) \circ l = l^{-1} \circ g_2 \circ l$
and by Math Induction from supposition
$h_n = l^{-1} \circ g_n \circ l$ obtain $h_{n+1} = h \circ h_n = \left(l^{-1} \circ g \circ l\right) \circ \left(l^{-1} \circ g_n \circ l\right) =$
$l^{-1} \circ (g \circ g_n) \circ l = l^{-1} \circ g_{n+1} \circ l$.
Since $f_n(x) = x \iff \left(l^{-1} \circ g_n \circ l\right)(x) = x \iff (g_n \circ l)(x) = l(x) \iff$
$g_n(l(x)) = l(x)$ then orbit $O_f(x)$ is n-periodic iff $O_g(l(x))$ is n-periodic.

Lemma 2.
For any Möbius Function $g(x) = \dfrac{ax+b}{cx+d}$ with $a, b, c, d \in \mathbb{R}$ and
$ad - bc \neq 0$, $c \neq 0$ there is linear function $l(x) = px + q$,such
that $h(x) = \left(l^{-1} \circ g \circ l\right)(x) = \dfrac{sign(ad-bc)}{t-x}$.
Proof.
Let $y = \dfrac{ax+b}{cx+d}$. We will find p, q such that

$$py + q = \frac{a(px+q)+b}{c(px+q)+d} \iff y = \frac{\pm 1}{t-x}.$$

$$py + q = \frac{a(px+q)+b}{c(px+q)+d} \iff py = \frac{a(px+q)+b}{c(px+q)+d} - q \iff$$

$$py = \frac{apx + aq + b - cpqx - cq^2 - dq}{cpx + cq + d} \iff$$

$$py = \frac{px(a-cq) + b + q(a-cq-d)}{cpx+cq+d}.$$

For $q = \dfrac{a}{c}$ we get $y = \dfrac{\dfrac{ad-bc}{(pc)^2}}{-\dfrac{a+d}{pc} - x}$ and by setting

$p := \dfrac{\sqrt{|ad-bc|}}{c}$ and $t := -\dfrac{a+d}{\sqrt{|ad-bc|}}$
we obtain $y = \dfrac{sign(ad-bc)}{t-x}$.
Corollary.
i. If $ad - bc > 0$ then g have $n-$periodic orbit iff
$-\dfrac{a+d}{\sqrt{ad-bc}} = 2\cos\dfrac{k\pi}{n}$, where $k = 1, 2, ..., n-1$ and k is coprime with n,
ii. If $ad - bc < 0$ then g always have $1-$periodic orbit; $2-$periodic orbit iff $a + d = 0$; and never $m-$periodical orbit for $m > 2$.

Part 5. Addition
In conclusion, we will consider a problem essentially similar
to those considered above, the solution of which demonstrates

a different approach.

Problem.

Let $n \geq 2$ be an integer.

Find all real numbers a such that there exist real numbers $x_1, ..., x_n$ satisfying

$x_1(1-x_2) = x_2(1-x_3) = ... = x_{n-1}(1-x_n) = x_n(1-x_1) = a.$

Solution.

Let A be set all real numbers a such that system of equations

$$\textbf{(4)} \begin{cases} x_k(1-x_{k+1}) = a, k = 1,2,..,n-1 \\ x_n(1-x_1) = a \end{cases}$$

is solvable with respect to $x_1, ..., x_n \in \mathbb{R}$.

Noting that for $a = 0$ the system **(4)** has obvious solution $x_1 = x_2 = ... = x_n = 0$ we assume further that $a \neq 0$.

That immediately implies that $x_i \neq 0, i = 1,2,...,n$ and we can rewrite the system as follows:

$$\textbf{(5)} \quad \begin{cases} x_{k+1} = h(x_k), k = 1,2,..,n-1 \\ x_1 = h(x_n) \end{cases}, \text{ where}$$

$h(x) := 1 - \dfrac{a}{x} = \dfrac{x-a}{x}.$

Let $h_1(x) := h(x), h_{n+1}(x) = h(h_n(x)), n \in \mathbb{N}$ and H_n be matrix of coefficients for Mobius function $h_n(x)$, that is

$h_n(x) = \dfrac{a_n x + b_n}{c_n x + d_n}$ and $H_n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}, n \in \mathbb{N}.$

Also let $h_0(x) := x$. Then $H_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $H_1 = H = \begin{pmatrix} 1 & -a \\ 1 & 0 \end{pmatrix}$ and

$$H_{n+1} = H \cdot H_n \iff \begin{pmatrix} a_{n+1} & b_{n+1} \\ c_{n+1} & d_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & -a \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} =$$

$$\begin{pmatrix} a_n - ac_n & b_n - ad_n \\ a_n & b_n \end{pmatrix} \iff \begin{cases} a_{n+1} = a_n - ac_n \\ b_{n+1} = b_n - ad_n \\ c_{n+1} = a_n \\ d_{n+1} = b_n \end{cases} \iff$$

$$\begin{cases} a_{n+1} = a_n - aa_{n-1} \\ b_{n+1} = b_n - ab_{n-1} \\ c_{n+1} = a_n \\ d_{n+1} = b_n \end{cases}, n \in \mathbb{N}$$

and $a_0 = 1, a_1 = 1, b_0 = 0, b_1 = -a.$

Since (a_n) and (b_n) satisfies to the same recurrence and $b_2 = -a$ then $b_n = -aa_{n-1}, n \in \mathbb{N}$.

Thus, $H_n = \begin{pmatrix} a_n & -aa_{n-1} \\ a_{n-1} & -aa_{n-2} \end{pmatrix}, n \geq 2$ and $h_n(x) = \dfrac{a_n x - aa_{n-1}}{a_{n-1} x - aa_{n-2}}, n \geq 2.$

Coming back to the system **(5)** we can see that

$x_k = h_k(x_1), k = 1,2,..,n-1$ and $x_1 = h_n(x_1)$,

that is x_1 is solution of equation $h_n(x) = x$.Thus $A_n = \{a \mid h_n(x) = x, x \in \mathbb{R}\}$.

Since $h_n(x) = x \iff \dfrac{a_n x - aa_{n-1}}{a_{n-1} x - aa_{n-2}} = x \iff$

$a_n x - aa_{n-1} = a_{n-1} x^2 - aa_{n-2} x \iff$

(6) $\quad a_{n-1}x^2 - x(a_n + aa_{n-2}) + aa_{n-1} = 0,$
where a_n is polynomial of a defined recursively by
$a_{n+1} = a_n - aa_{n-1}, n \in \mathbb{N}$, $a_0 = 1, a_1 = 1$
and quadratic equation **(6)** is solvable in real x iff its discriminant
$D_n := (a_n + aa_{n-2})^2 - 4aa_{n-1}^2 \; = a^2a_{n-2}^2 + 2aa_na_{n-2} - 4aa_{n-1}^2 + a_n^2 =$
$a^2a_{n-2}^2 - 4aa_{n-1}^2 + 2aa_{n-2}(a_{n-1} - aa_{n-2}) + (a_{n-1} - aa_{n-2})^2 =$
$a_{n-1}^2(1-4a) = a_{n-1}^2(1-4a)$ is non negative then

$$A_n = \left\{a \mid a_{n-1}^2(1-4a) \geq 0\right\} = (-\infty, 1/4] \cup \{a \mid a_{n-1} = 0\}, n \geq 2.$$

For example,
$a_2 = 1 - a, a_3 = 1 - 2a, , a_4 = a^2 - 3a + 1, a_5 = a^2 - 3a + 1 - a(1-2a) =$
$3a^2 - 4a + 1$ and $A_2 = (-\infty, 1/4],\; A_3 = (-\infty, 1/4] \cup \{1\},$

$$A_4 = (-\infty, 1/4] \cup \{1/2\}, A_5 = (-\infty, 1/4] \cup \left\{\frac{3-\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}\right\}.$$

Note that for any $a \leq \dfrac{1}{4}$ system **(1)** solvable in $\mathbb{R}$.
Indeed, since

$$h(x) = x \iff x^2 - x + a = 0 \iff x \in \left\{\frac{1-\sqrt{1-4a}}{2}, \frac{1+\sqrt{1-4a}}{2}\right\}$$

then $(x_1, x_2, ..., x_n) = (x, x, x, ..., x)$ for any such x
is solution of **(1)** because for $x_1 = x$ we have
$h_k(x_1) = h_k(x) = x, k = 1, 2, ..., n.$
Therefore, to complete the solution of the problem remains find
all solution of equation $a_{n-1}(a) = 0$ in real $a > 1/4$ for any $n \geq 2$.
Since $a > 1/4 \iff \dfrac{1}{2\sqrt{a}} < 1$ then denoting

$$\alpha := \arccos\frac{1}{2\sqrt{a}} \text{ and } b_n := \frac{a_n}{(\sqrt{a})^n} \text{ we obtain}$$

$$a_{n+1} = a_n - aa_{n-1} \iff \frac{a_{n+1}}{(\sqrt{a})^{n+1}} - \frac{1}{\sqrt{a}} \cdot \frac{a_n}{(\sqrt{a})^n} + \frac{a_{n-1}}{(\sqrt{a})^{n-1}} = 0 \iff$$

(4) $\quad b_{n+1} - 2\cos\alpha \cdot b_n + b_{n-1} = 0, n \in \mathbb{N}.$
Since $b_n = c_1\cos n\alpha + c_2\sin n\alpha$ and $b_0 = 1, b_1 = \dfrac{1}{\sqrt{a}} = 2\cos\alpha$
we obtain $c_1 = 1, c_2 = \cot\alpha$ and, therefore,

$$b_n = \cos n\alpha + \cot\alpha\sin n\alpha = \frac{\sin(n+1)\alpha}{\sin\alpha}, n \in \mathbb{N}.$$

Thus, for any $n \geq 2$ we have

$$a_n = \frac{a^{n/2}\sin(n+1)\alpha}{\sin\alpha} \text{ and } a_n = 0 \iff \begin{cases} \sin(n+1)\alpha = 0 \\ \sin\alpha \neq 0 \\ a = \dfrac{1}{4\cos^2\alpha} \end{cases} \iff$$

$$\begin{cases} \alpha = \dfrac{1}{4\cos^2\dfrac{k\pi}{n+1}} \\ k = 1, 2, ..., n \\ a = \dfrac{1}{4\cos^2\alpha} \end{cases} \iff a = \frac{1}{4\cos^2\dfrac{k\pi}{n+1}}, k = 1, 2, ..., \left\lfloor\frac{n+1}{2}\right\rfloor$$

(since $\cos^2 \dfrac{k\pi}{n+1} = \dfrac{(n+1-k)\,\pi}{n+1}$, $k = 1, 2, .., n$).
Thus, for any $n \geq 2$ equation $h_n(x) = x$ solvable in $\mathbb{R}$ Iff

$$a \in A_n = (-\infty, 1/4] \cup \left\{ \frac{1}{4\cos^2 \dfrac{k\pi}{n}} \mid k = 1, 2, ..., \left\lfloor \frac{n}{2} \right\rfloor \right\}.$$

Remark.
Of course, this problem also can be solved by following the instructions that represented in Generalization 3 and realize this opportunity we we will leave to readers.